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HARMONIC MAPS, HYPERBOLIC COHOMOLOGY AND HIGHER MILNOR INEQUALITIES

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A. PRELIMINARY RESULTS

A.1. THEOREM. (*Milnor* [14]). *Let* $\pi:\pi_1(\Sigma^g)\to SL_2(\mathbb{R})$ *be a representation of the surface group and let E be the corresponding flat vector bundle of rank two over* Σ^g. *Then the Euler number* $(\chi(E),[\Sigma])$ *satisfies*

$$|(\chi(E)),[\Sigma])|\le g-1.$$

A.2. THEOREM. (*Goldman* [6]). *Let* $\pi:\pi_1(\Sigma^g)\to PSL_2(\mathbb{R})$ *be a representation and let* ξ *be the associated* S^1*-bundle over* Σ. *Then* $\chi(\xi)\le 2g-2$ *and if the equality holds, then the image of* π *acts discontinuously and cocompactly in the hyperbolic plane* $\mathscr{H}^2$.

A.3. THEOREM. (*Thurston* [20]). *Let M be a compact hyperbolic manifold. Then for any continuous map* $f:\Sigma^g\to M$ *there exists a smooth map* $\bar{f}$, *homotopic to f, such that* $Area(\bar{f})\le 4\pi(g-1)$.

A.4. THEOREM. (*Sullivan* [19]). *Let* M^n *be a triangulated manifold with precisely* d_n *n-simplices, and let E be a flat vector bundle of rank n over M. Then*

$$|\chi(E),[M]|\le d_n.$$

A.5. *Remark.* (Lusztig, see Gromov [7], p. 22) For any compact manifold M, and any real Lie group G, there are no more than a finite number of flat G-bundles over M, nonisomorphic as bundles (without connection).

A.6. THEOREM. *Kapovich* [12]). *Let* M^4 *be a complete hyperbolic manifold and let* Σ^{g_1}, Σ^{g_2} be two singular incompressible surfaces in M. Then

$$|[\Sigma^{g_1}]\cap[\Sigma^{g_2}]|\le C(g_1,g_2)$$

for some universal function C.

A.7. THEOREM. (*Gromov* [8], p. 145). *Let M be a compact manifold of negative curvature, and let* π *be a finitely presented group which is not a free product. Then there are at most finite number of embeddings* $f:\pi\to\pi_1(M)$ *up to a conjugation by an element of* $\pi_1(M)$ *and an automorphism of* π.

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B. INTRODUCTION

The celebrated Milnor inequality A.1. gives necessary and sufficient conditions for a plane bundle over a surface to carry a flat connection. The reader will find a discussion of various generalizations in spirit of the Sullivan's theorem in the paper of Gromov [7], pp. 21–24. In the other direction, Goldman [6] showed that a deep connection holds between the Milnor inequality and the hyperbolic geometry. He associated to a representation of a surfaces group $\pi_1(\Sigma^g)$ in $PSL_2(\mathbb{R})$ a *flat* $\mathscr{H}^2$*-bundle* and then constructed a developing section using a clever induction argument. Later Hitchin [10] showed that the Goldman's theorem may be approached from the study of self-duality equations on Σ^g. In the subsequent paper, Donaldson [5], proved that these equations describe harmonic sections of the underlying flat $\mathscr{H}^3$-bundle. Such sections, also called twisted harmonic maps, were intensively used by Corlette [3] and Jost-Yau [11] for proving rigidity results of Margulis type.

One of the objectives of this work is to give a new and very simple proof of the Goldman's theorem A.2. and, therefore, also the Milnor inequality A.1., using the sharp version of the Thurston inequality A.3. in the case of harmonic sections. This version, given in C.1. below, gives a sharp estimate for variable curvature of the ambient space (comp. Gromov [8] p. 145).

Moreover, we give a new proof of the Toledo inequality [21, 22] for the first Chern class of a negative subbundle of a flat $SU(1, n)$-bundle over Σ^g, with a slightly weaker constant. Our method works for representations of $\pi_1(\Sigma^g)$ is the isometry group of any Kählerian manifold of negative curvature.

Next, we pass to a flat $SO(1, n)$-bundle over a manifold M. We study a secondary characteristic class in $H^*(M, \mathbb{R})$, which coincides with the Euler class of any negative subbundle if n is even. This secondary characteristic class is a sort of "hyperbolization" of the "differential characters" of Cheeger–Simons [2]. We use the Thurstion straightening technique to give effective estimates for its value on [M] when dim $M = n$.

We introduce then a secondary characteristic class for a flat $Sp(1,n)$-bundle over M. A somewhat more sophisticated straightening technique allows us to give effective estimates of it in terms of a given triangulation of M.

The last part of the paper deals with the application of the Thurston inequality and some isoperimetrical results to "hyperbolic cohomology", i.e. to deriving restrictions on topology of negatively curved manifolds. Those concerning the multiplicative structure in cohomology are especially interesting. A recent theorem of Kapovich, A.6., gives such restrictions for the case of constant negative curvature. We prove formally similar results for compact manifolds of variable negative curvature below in F.1. Moreover, using the technique of Schoen-Yau [16], we prove the version of the finiteness theorem A.7. when π is the surface group $\pi_1(\Sigma^g)$.

C. SHARP THURSTON INEQUALITY

Let N be a compete Riemannian manifold with the curvature satisfying $-K \leq K(N) \leq -k < 0$. Let ISO(N) be the isometry group of N. Consider a flat N-bundle $N \to E \to \Sigma^g$ over a closed surface Σ^g, whose holonomy group lies in ISO(N). For a section $\sigma\colon \Sigma^g \to E$ one defines its area, Area(σ) using local projections to fibers, i.e. considering (locally) σ as a map from Σ^g to N. Suppose now that N is simply connected and the natural action of the holonomy subgroup of ISO(N) on the sphere at infinity $S_\infty(N)$ is fixed-point-free. Then we have a following result.

C.1. THEOREM. *There exists a section, σ, satisfying*

$$\mathrm{Area}(\sigma) \leq \frac{4\pi(g-1)}{k}.$$

For trivial bundles and compact N, the results is modified as follows.

C.2. THEOREM. *Let $f\colon \Sigma^g \to M$ be a smooth map to a compact Riemannian manifold, whose curvature satisfies $-K \leq K(M) \leq -k < 0$. Then there exists a smooth map $\bar{f}$ homotopic to f, such that*

$$\mathrm{Area}(\bar{f}) \leq \frac{4\pi(g-1)}{k}.$$

C.3. *Remark*: See Thurston [20] for the proof in the hyperbolic case ($k = K$). We will first prove the theorem C.2. and then show how to modify the argument to deal with the twisted conditions of C.1.

Proof of C.2. First we fix a metric, say h_Σ, on Σ of constant negative curvature, and denote by h_M the given metric of M. By the existence theorem of Alber–Eells–Sampson, there exists a harmonic map $\bar{f}\colon (\Sigma, h_\Sigma) \to (M, h_M)$, homotopic to f. Consider the product $\hat{M} = \Sigma \times M$ with the metric $\varepsilon h_\Sigma + h_M$, $\varepsilon > 0$. Let $\varphi\colon \Sigma \to \hat{M}$ be the graph of $\bar{f}$, i.e. $\varphi = (id, \bar{f})$, so φ is a harmonic embedding of Σ. We need a following lemma.

C.4. LEMMA. *Let $\varphi\colon \Sigma \to N$ be a harmonic immersion. For $x \in \Sigma$, let $A_{\varphi(x)}$ be the second quadratic form of $\varphi(\Sigma)$ at $\varphi(x)$. Then*

$$Tr_{h_\Sigma}(A_{\varphi(x)} \circ D\varphi) = 0.$$

Proof of C.4. Let $v(x)$ be any normal vector field to φ, and let $\mu(x)$ be any smooth function. Consider a variation $\varphi_t(x)$ such that $\frac{d}{dt}\varphi_t(x) = \mu(x)v(x)$. By the first variation formula we get $0 = \frac{d}{dt}$ energy $(\varphi_t(x)) = 2\int \mu(x)\, Tr_{h_\Sigma}(A_v \circ D_\varphi) = 0$, and the result follows.

Next, it follows from C.4. that $\det A_{\varphi(x)} \leq 0$ for all x, so by the Gauss–Bonnet we have

$$4\pi(g-1) \geq -\int_\Sigma K_{\hat{M}}(T_{\varphi(x)}\varphi(\Sigma))\, d\mathrm{area}_{\hat{M}}.$$

We claim that the last integral majorises $-\int_{\tilde{\Sigma}} K_M(T_{\bar{f}(x)}\bar{f}(\Sigma))\, d\mathrm{area}_M$, as $\varepsilon \to 0$, where $\tilde{\Sigma} \subset \Sigma$ consists of those points where $D\bar{f}$ has the maximal rank two. Indeed, $-K_{\hat{M}}$ is everywhere nonnegative and it is clear that locally in $\tilde{\Sigma}$, $K_{\hat{M}}(T_{\varphi(x)}\varphi(\Sigma))$ goes to $K_M(T_{\bar{f}(x)}\bar{f}(\Sigma))$ as $\varepsilon \to 0$. Finally, we get

$$4\pi(g-1) \geq -\int_{\tilde{\Sigma}} K_M(T_{\bar{f}(x)}\, \bar{f}(\Sigma)) \geq k \cdot \mathrm{Area}(\bar{f}),$$

proving C.2.

Proof of C.1. We use the Donaldson existence theorem [5] instead of the Alber–Eells–Sampson theorem, and find a harmonic section of Σ. The further computations are just the same as in the Proof of C.2.

We are now in position to prove the Goldman theorem A.2.

C.5. *Proof of the Theorem* A.2. Consider the associated flat $\mathscr{H}^2$-bundle $\mathscr{E}$ over Σ. We can assume that the action of $\pi_1(\Sigma^\delta)$ on S^1 does not have fixed points, otherwise $\chi(\xi) = 0$, so we can apply the Theorem C.1. and find a section σ of $\mathscr{E}$ with Area$(\sigma) \leq 4\pi(g-1)$. We next consider over $\mathscr{E}$ the vertical bundle E (tangent to fibers). Fit together the Levi–Civita connection along the fibers and the flat connection of $\mathscr{E}$ to get a connection in E. Its curvature form is just the inverse image of the area form on fibers under the locally well-defined projections to fibers. So by Chern–Weil we have

$$2\pi |\chi(E|_s)| \leq \mathrm{Area}(s)$$

for any section s. Now notice that since $\mathscr{H}^2$ is a cell, the left side is independent of s and is always equal to $2\pi|\chi(\xi)|$. So choosing s as above, we get $|\chi(\xi)| = |\chi(E|_s)| \leq \frac{1}{2\pi}\mathrm{Area}(s) \leq 2g - 2$. If the equality holds, then the Jacobian of s (defined locally viewing s as a map from Σ to $\mathscr{H}^2$) does not change sign. Since s is harmonic, we can apply the argument of Schoen–Yau [17, pp. 270–271] which implies that actually rank $Ds = 2$ everywhere. Hence s induces a hyperbolic structure on Σ and it is clear that the corresponding representation of $\pi_1(\Sigma^\delta)$ in $\mathrm{PSL}_2(\mathbb{R})$ is just the holonomy of $\mathscr{E}$ which is π. This proves A.2.

D. TOLEDO INEQUALITY FOR FLAT $SU(1, n)$-BUNDLES

D.1. Before we shall proceed any further we give yet another reformulation of the Milnor and Goldman theorems A.1., A.2., which emphasize the role of the structure group. Consider a representation $\pi_1(\Sigma^g) \to SO(1, 2)$ and the correspondent flat vector bundle of rank 3 with a self-parallel metric of the signature (1.2). Let E_- be a (unique up to equivalence) negative subbundle of E, i.e. a subbundle of E such that the restriction of the metric on it in negatively definite. Then $\chi(E_-)$ is an invariant of the representation and the Milnor–Goldman inequality says that

$$|\chi(E_-), [\Sigma]| \leq 2g - 2.$$

This formulation suggests that, for structure groups others than $SO(1, 2)$, a similar result may hold. This is so indeed for the pseudoorthogonal group $SO(1, n)$, as we will see in E.9. below. For the pseudounitary structure group, $SU(1, n)$, one has the following result, [22], which we will prove with a weaker constant.

D.2. THEOREM. *Let $\pi:\pi_1(\Sigma^\delta) \to SU(1,n)$ be an irreducible representation in $\mathbb{C}^{m+1}$, and let E be the corresponding flat complex vector bundle with the self-parallel Hermitian metric of the signature $(1, n)$. Then for a negative subbundle E_- one has*

$$|c_1(E_-)| \leq (g - 1).$$

D.3. *Remark.* Since E is flat all $c_i(E)$ vanish. So for a positive line subbundle E_+ one has $|\deg E_+| = |c_1(E_-)|$.

D.4. *Proof of* D.2. Consider the unit ball B^n in $\mathbb{C}^n$ with the Bergman metric and $SU(1, n)$ acting isometrically and construct a flat B^n-bundle $\mathscr{F}$ over Σ, associated to π. For a section s of $\mathscr{F}$ let $F|s$ be the restriction of the vertical bundle F over $\mathscr{F}$, on s.

LEMMA. $|c_1(F|s)| = 2|c_1(E_-)|$.

Proof. Consider the subfibration of E, say G, consisting of those vectors in fibers, whose length (with respect to the selfparallel Hermitian metric) is 1. Let E_+ be the orthogonal complement to E_- in E. We give a realization of $\mathscr{F}$ as a quotient G/S^1 under the action of $S^1 \subset \mathbb{C}$ by multiplication. For a local unit section z of E_+ we can write the equation of G as

$$|\alpha|^2 - |z_-|^2 = 1,$$

where $\alpha z + z_- \in G, z_- \in E_-$. In G/S^1 we can always choose a representative such that

$\alpha = 1$, hence identifying G/S^1 with E_-. When z is changed to $e^{i\beta}z$, where β is a smooth real function, this identification will be twisted by $e^{i\beta}$. Thus for a section σ of G,

$$F|\sigma \approx \varepsilon^* E_-|\sigma \otimes \varepsilon^* E_+^*|\sigma$$

where $\varepsilon:\mathscr{F} \to \Sigma$ is the bundle map. So $c_1(F|\sigma) = c_1(E_-|\sigma) - c_1(E_+|\sigma) = 2c_1(E_-)$ by D.3. (we usually identify Σ and $\sigma(\Sigma)$). Now we extend the Bergman metric of B^n to all fibers of $\mathscr{F}$, using the flat connection. We claim the action of the holonomy group in the sphere at infinity is fixed-point-free. Indeed, the space of geodesics of B^n identified with G/S^1 is just the quotient of the space of all Lagrangian two-planes in $\mathbb{C}^{n+1}$, such that the restriction of the Hermitian form on them is not definite, under the natural action of S^1. Every such plane contains precisely two isotropic lines, so the sphere at infinity is $K/\mathbb{C}^*$, where K is the isotropic cone of $\mathbb{C}^{n+1}$. So if the action of the holonomy group were not fixed-point-free, the initial representation π would be reducible. That means we are able to apply the existence theorem of Donaldson and find a harmonic section $\sigma:\Sigma \to \mathscr{F}$.

For the following computation consider the complex structure J of B as a section of $\Lambda^2 T_{\mathbb{R}}B$. Then the value of the Kähler form on an element z of $\Lambda^2 T_{\mathbb{R}}B$, is just (J, z), whereas the curvature form acts as $-R_B(z) = (z, z) + (J, z)^2$. For a section $\sigma(x)$ denote $z(x)$ the unit vector in $\Lambda^2 T_{\sigma(x)}\mathscr{F}_x$, representing the tangent space of the corresponding (locally defined) surface in B. By Chern–Weil we have

$$c_1(F|\sigma) = \int_\sigma \frac{3}{2\pi}(J, z).$$

From the proof of C.2. we get

$$4\pi(g-1) \geq \int ((z, z) + (J, z)^2)\, d\text{area}$$

so

$$\int |(J, z)|\, d\text{area} \leq \sqrt{\int (J, z)^2\, d\text{area} \cdot \text{Area}(\sigma)} \leq \frac{1}{2}\int ((J, z)^2 + (z, z))\, d\text{area} \leq 2\pi(g-1)$$

and, finally,

$$|c_1(E_-)| = \frac{1}{2}|c_1(F|\sigma)| = \frac{3}{4\pi}\left|\int_\sigma (J, z)\right| \leq \frac{3}{2}(g-1).$$

E. SECONDARY CHARACTERISTIC CLASSES FOR REPRESENTATIONS IN $SO(1, n)$ AND $Sp(1, n)$

E.1. Let M be a compact manifold and let $\pi:\pi_1(M) \to SO(1, n)$ be a representation. Denote by $\mathscr{F}$ the corresponding flat $\mathscr{H}^n$-bundle over M. Let ω be the volume form of $\mathscr{H}^n$, lifted to $\mathscr{F}$, and let σ be any section of $\mathscr{F}$.

Definition. The volume class $\text{Vol}(\pi) \in H^n(M, \mathbb{R})$ is defined as

$$\text{Vol}(\pi) = \sigma^*\omega.$$

E.2. *Independence.* Since $\mathscr{H}^n$ is a cell, all sections of $\mathscr{F}$ are homotopic to each other, so $\text{Vol}(\pi)$ is a well-defined invariant of π.

E.3. *Even Dimension.* Let E be the flat vector bundle, associated to π and let E_- be any negative subbundle of E. If n is even, then

$$\text{Vol}(\pi) = \frac{\text{Vol}(S^n)}{2}\chi(E_-).$$

Proof. Following D.4., one realizes $\mathscr{F}$ as a (hyperboloid) subfibration on E, and for the vertical bundle F one gets $F|\mathscr{s} \approx E_-$. The formula then follows from the Gauss–Bonnet–Weyl formula for χ.

E.4. $\mathrm{Vol}(\pi)$ and "Differential Characters". The invariant $\mathrm{Vol}(\pi)$ can be looked at as a "hyperbolic version" of the secondary characteristic classes introduced by Cheeger and Simons in [2]. Their classes, called "differential characters" do not lie in the cohomology ring, basically because the fiber of the bundle, considered by Cheeger and Simons is a sphere, and so we do not have the independence property E.2.

E.5. *Functoriality.* For a continuous map $f: M' \to M$ and a representation $\pi: \pi_1(M) \to SO(1,n)$ one has $\mathrm{Vol}(\pi \circ f_*) = f^* \mathrm{Vol}(\pi)$. In particular, if $\dim M' = \dim M = n$, then $(\mathrm{Vol}(\pi \circ f_*), [M']) = \deg f \cdot (\mathrm{Vol}(\pi), [M])$.

E.6. *Stability.* Let $\mathrm{Rep}(\pi_1(M), SO(1,n))$ be the representation variety of the fundamental group of M. Then the map $\mathrm{Vol}: \mathrm{Rep}(\pi_1(M), SO(1,n)) \to H^n(M, \mathbb{R})$ is locally constant, if n is even.

Proof. Use E.3. and the fact that the isomorphism class of E_- is stable when π ranges in a connected component of $\mathrm{Rep}(\pi_1(M), SO(1,n))$.

E.7. *Example.* Let M be a hyperbolic manifold, and let $\pi: \pi_1(M) \to SO(1,n)$ be the fundamental representation. Then

$$(\mathrm{Vol}(\pi), [M]) = \mathrm{Vol}(M).$$

E.8. Let $BSO^\delta(1,n)$ be the classifying space of $SO(1,n)$ made discrete. Then the universal volume class is the element in $H^n(BSO^\delta(1,n), \mathbb{R})$, which is in the image of the natural map $H^*_{\mathrm{cont}}(SO(1,n)) \to H^*(BSO^\delta(1,n))$. It is hyperbolic in the following sense.

Let $\mathscr{K}$ be a triangulation of M with precisely d_n n-dimensional simplices, where $n = \dim M$. Remark that continuous maps from M to $BSO^\delta(1,n)$ are in one-to-one correspondence with representations of $\pi_1(M)$ is $SO(1,n)$, up to homotopy.

E.9. THEOREM. *For any representation $\pi: \pi_1(M) \to SO(1,n)$ and any triangulation $\mathscr{K}$,*

$$(\mathrm{Vol}(\pi), [M]) \leq \mu_n d_n,$$

where μ_n depends only on n.

Proof. The idea to prove E.9., and the more complicated case of variable curvature in E.11. below, is to use a twisted version of the Thurston straightening process. We construct a special section $\mathscr{s}$ of $\mathscr{F}$ as follows. First choose it arbitrarily over the 0-skeleton. Given a 1-simplex, say $\sigma^1 = xy$ in k, trivialize the bundle $\mathscr{F}|\sigma^1$ to be $\sigma^1 \times \mathscr{H}^n$ and join $\mathscr{s}(x)$ and $\mathscr{s}(y)$ by the unique shortest geodesic. This gives the extension of $\mathscr{s}$ to the 1-skeleton. Next, given a 2-simplex $\sigma^2 = xyz$, trivialize $\mathscr{F}|\sigma^2 \approx \sigma^2 \times \mathscr{H}^n$, so that $\mathscr{s}|\partial\sigma^2$ becomes a geodesic triangle, and fill it in the totally geodesic plane $\mathscr{H}^2$ spanned by $\mathscr{s}|\partial\sigma^2$. We proceed in this way, and construct a section $\mathscr{s}$, such that for any simplex σ^k, $\mathscr{s}|\sigma^k$ maps σ^k to a geodesic simplex in $\mathscr{F}|\sigma^k \approx \sigma^k \times \mathscr{H}^n$. Then we see that $(\mathrm{Vol}(\pi), [M]) \leq \mu_n d_n$, where μ_n is the Milnor constant, i.e. the maximal volume of a n-simplex in $\mathscr{H}^n$.

E.10. *Definition.* Let $\pi: \pi_1(M) \to Sp(1,n)$ be a representation and let E be the corresponding flat quaternoinic vector bundle of the rank $n+1$. Consider a negative subbundle E_-, with respect to the self-parallel quaternionic Hermitian form. Then the volume $\mathrm{Vol}(\pi)$ is defined as the Euler class $\chi(E_-)$.

E.11. THEOREM. *For any triangulation k of M and a representation $\pi:\pi_1(M)\to Sp(1,n)$ one has*

$$(\mathrm{Vol}(\pi),[M])\le C_n\cdot d_n,$$

where C_n depends only on n.

E.12. LEMMA. *Let N^n be a simply connected complete Riemannian manifold of the negative curvature satisfying $-K\le k(N)\le -k$. Let $\pi:\pi_1(M)\to ISO(N)$ be a representation and let $\mathscr{F}$ be the associated flat N-bundle over M. Then there exists a section s of $\mathscr{F}$ satisfying* $\mathrm{Vol}(s)\le c(n,k,K)\cdot d_n$.

E.13. *Proof of the Theorem* E.11. Let N be the queternionic hyperbolic space with the isometrical action of $Sp(1,n)$, and let $\mathscr{F}$ be the corresponding flat N-bundle. We consider a realization of $\mathscr{F}$ out of the flat $\mathbb{H}$-vector bundle E with the self-parallel quaternionic Hermitian form of the signature $(1,n)$, as follows. Let G be the subfibration of those vectors in E, whose length is equal to 1. The group S^3 of unit quaternions acts on G and we put $\mathscr{F}=G/S^3$. Then the computation, analogous to D.4. shows that $F|s\approx E_+^*\otimes_{\mathbb{H}}E_-$ as real vector bundles over a section s. Here we identify $E_\pm$ with their lift on $S(M)$. Let $\mu\in H^4(M)$ be the Euler class of E_+.

E.14. LEMMA. $\chi(F)=\pm(n+1)\mu^n=\pm(n+1)\chi(E_-)$.

Proof. Fix $J\in S^3$ with $J^2=-1$ and consider all $\mathbb{H}$-vector bundles to be complex bundles with respect to J. Then $c_1(E_+)=0$ since $S^3\approx SU(2)$, and $c_2(E_+)=\mu$. By the classifying space argument we may check $\chi^2(F)=(n+1)^2\mu^2$ instead of $\chi(F)=(n+1)\mu$. Passing to complexifications, we get $F_{\mathbb{C}}\approx E_+^*\otimes_{\mathbb{C}}E_-$, so $(\chi(F))^2=c_{4n}(F_{\mathbb{C}})=c_{4n}(E_+^*\otimes E_-)$. We may assume $E_+^*=L\otimes L^*$ for some line bundle L, using $c_1(E_+)=0$ and the splitting principle. Then $E_+^*\otimes E_-=L\otimes E_-\otimes L^*\otimes E_-$, so $c(E_+^*\otimes E_-)=\sum_{i=0}^{2n}c_i\lambda^{2n-i}\times\sum_{j=0}^{2n}(-1)^jc_j\lambda^{2n-j}=\sum_{k=0}^{2n}\sum_{i=0}^{k}c_ic_{2k-i}\lambda^{4n-2k}$, where $\lambda=c_1(L)$ and $c_i=c_i(E_-)$. But since E is flat, $1=c(E)=c(E_+)c(E_-)=(1-\mu)c(E_-)$ so $c_{\mathrm{odd}}(E_-)=0$ and $c_{2s}(E_-)=\mu^s$, so $c_{4n}(E_+^*\otimes_{\mathbb{C}}E_-)=(n+1)^2\mu^{2n}$ and $\chi(F)=\pm(n+1)\mu^n$.

So we can replace the estimate of $\chi(E_-)$ to that of $\chi(F)$. But the quaternionic hyperbolic space N has negative curvature, so we can apply Lemma E.12. to complete the proof of the Theorem E.11.

Proof of the Lemma E.12. We start with the choice of a section over the 0-skeleton of k in an arbitrary way. Then, given a 1-simplex, say $\sigma^1=xy$, we trivialize $\mathscr{F}|\sigma^1$ as in E.9 and join $s(x)$ and $s(y)$ by the shortest geodesic in N. Next, given a 2-simplex, $\sigma^2=xyz$, we trivialize $\mathscr{F}|\sigma^2\approx\sigma^2\times N$ so that $S|\partial\sigma^2$ becomes a geodesic triangle. We find a minimal bubble Δ spanning $S|\partial\sigma^2$ by the solution to Plateau problem. Since the curvature of N is negative, we have (comp. Gromov [7]) $k\mathrm{Area}(\Delta)\le\pi-\measuredangle x-\measuredangle y-\measuredangle z<\pi$, so $\mathrm{Area}(\Delta)\le\frac{\pi}{k}$. We extend s to the interior of σ^2 using Δ. So we get the extension to the second skeleton. Now, let σ^3 be a three-simplex and let $p\in\mathrm{int}\sigma^3$. We assume $\mathscr{F}|\sigma^3\approx\sigma^3\times N$. Choose any point in N to stand for $s(p)$. Then we triangulate σ^3 baricentrically from $s(p)$. For any 3-simplex of the subdivision, say $p\sigma^2$, we put $s|p\sigma^2$ to be the geodesic cone over $s|\sigma^2$ from $s(p)$. The linear isoperimetrical inequality [7] then gives $\mathrm{vol}(s|\sigma^3)\le\mathrm{const}(k,K)$. We proceed inductively in this way and construct a section with $|(\mathrm{Vol}(s),[M])|\le\mathrm{const}(n,k,K)d_n$, providing E.12.

E.15. *Remark.* One can use the Gromov's simplicial volume invariant [7] instead of d_n in the theorem E.11.

F. THE THURSTON INEQUALITY AND THE HYPERBOLIC COHOMOLOGY

F.1. THEOREM. *Let M^4 be a compact manifold of the negative curvature satisfying*

$$-K \leq K(M) \leq -k.$$

Let $\Sigma^{g_1}, \Sigma^{g_2}$ be two singular surfaces in M, i.e. two continuous maps from the surfaces of genera g_1, g_2 respectively, to M. Then for the elements $[\Sigma^{g_i}] \in H_2(M, \mathbb{Z})$ one has $[\Sigma^{g_1}] \cap [\Sigma^{g_2}] \leq C(g_1, g_2, k, K, \chi(M))$ for some functon C.

The result for hyperbolic M, not necessarily compact, and incompressible surfaces Σ^{g_i}, is due to Kapovich [12]. Our proof is analytic, whereas Kapovich [12] uses geometrical arguments from Thurston theory to provide a "canonical form" for a surface in a hyperbolic M^4.

F.2. LEMMA. *Let $\omega \in \Omega^2(M)$ be a harmonic form. Then* $\|\omega\|_{L^\infty} \leq \mathrm{const}(k, K, \chi(M)) \|\omega\|_{L^2}$.

Proof. By a theorem of P. Li [13] one has $\|\omega\|_{L^\infty} \leq c_1(k, K, \mathrm{Vol}(M), \mu(M)) \|\omega\|_{L^2}$, where $\mu(M)$ is the Sobolev constant. The result of Croke [4] shows $\mu(M) \geq c_2(\mathrm{Vol}(M), \mathrm{diam}(M), k, K)$.

By a theorem of Gromov [9], $\mathrm{diam}(M) \leq c_3(\mathrm{Vol}(M))$, and, finally, the Chern theorem gives $c_4(k, K)\chi(M) \leq \mathrm{Vol}(M) \leq c_3(k, K)\chi(M)$, which competes the proof.

Proof of the Theorem F.1. First we find surfaces $\tilde{\Sigma}^{g_1}, \tilde{\Sigma}^{g_2}$, homotopic to Σ^{g_1}, Σ^{g_2} respectively and such that $\mathrm{Area}(\Sigma^{g_i}) \leq \frac{4\pi(g_i - 1)}{k}$ using C.2. Next, let $\omega_1, \dots, \omega_N$ be the orthonormed basis of harmonic 2-forms on M. By F.2. we have $(\omega_i, \Sigma^{g_i}) \leq c(k, K, \chi(M)) \cdot g_i$. The Poincaré duality operator $D: H_2(M, \mathbb{R}) \to H^2(M, \mathbb{R})$ acts as folllows: $D[\Sigma^{g_j}] = \sum_i (\omega_i, [\Sigma^{g_j}])[\omega_i]$. Hence $|[\Sigma^{g_1}] \cap [\Sigma^{g_2}]| = |D[\Sigma^{g_1}] \cup D[\Sigma^{g_2}], [M]| = |\sum_i (\omega_i, [\Sigma^{g_1}]) \cdot (\omega_i, [\Sigma^{g_2}])| \leq c(k, K, \chi(M)) \cdot N \cdot g_1 \cdot g_2$. Moreover, by the Gromov finiteness theorem [9], or by the Betti number estimate [7], $N = b_2(M)$ is bounded by $\chi(M)$, so $|(\Sigma^{g_1}] \cap [\Sigma^{g_2}]| \leq c(k, K, \chi(M))$ as promised.

F.3. COROLLARY. *Suppose M is a surface fibration over a surface: $\Sigma^{g_1} \to M \to \Sigma^{g_2}$, possessing a section $\mathfrak{o}$. Let v be the Euler number of the vertical bundle over $\mathfrak{o}$. If*

$$|v| \geq c(K/k, g_1, g_2)$$

then there does not exist a metric of negative curvature between k and K, on M.

Remark 1.2. For complete metrics on vector bundls over Σ^{g_2} and zero section $\mathfrak{o}$, this becomes a conjecture of Kapovich [12]. See also Anderson [1] for positive results. For the case of holomorphic line bundles over Kählerian surfaces which admit a complex hyperbolic structure, see [15].

The following result and the theorem A.7., first appeared in Thurston [20] and Gromov [8] with a proof sketched there. Further discussion is in Sela and Rips [18]. Notice that A.7. deals with the groups more general than a surface group.

F.4. THEOREM. *Let M be a compact manifold of negative curvature and let $\pi_1(\Sigma^g)$ be the surface group. There exist no more than a finite number of embeddings $S: \pi_1(\Sigma^g) \to \pi_1(M)$ up to conjugations by an element of $\pi_1(M)$ and an automorphism of $\pi_1(\Sigma^g)$.*

Proof. Suppose $\varphi_i: \pi_1(\Sigma^g) \to \pi_1(M)$ is a sequence of mutually nonconjugate embeddings. Consider a sequence of continuous maps $f_i: \Sigma^g \to M$ with $f_{i*} = \varphi_i$. By the existence theorem for harmonic maps, we can choose f_i to be harmonic with respect to some metric of curvature -1, say h, on Σ^g. Then by the theorem C.2. we have Area$(f_i) \leq$ const. Now choose a conformal structure, say $\mathscr{G}_i$ on Σ, such energy $\mathscr{G}_i(f_i) \leq 2$Area$(f_i) \leq$ const. Let H_i be the unique metric of the curvature -1 in $\mathscr{G}_i$. The argument of Schoen and Yau [16] shows that the class of H_i in the modular space M_{6g-6} is contained in some compact subset. So, twisting if necessary by a diffeomorphism of Σ, we can assume that Energy$_n(f_i) \leq$ const with respect to some fixed metric h of the curvature -1. Let γ be a closed geodesic of H. Applying again the Shoen and Yau arguments, we get that length $f_i(\gamma)$ remains bounded as $i \to \infty$, which is a contradiction to the nonconjugacy of $\mathscr{G}_i$, and the theorem follows.

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Added December 1992. The reader will find the further study of "hyperbolic elements" in cohomology of various manifolds and spaces in the recent papers of the author:

Arithmetics and geometry of the Chern–Simons invariant, I, II, preprints,

Cyclic calculus and the non-commutative geometry of Connes, preprint.

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